Linear System Theory – lecture 7

Obs. & Ctrl. part 2.

Ph. Müllhaupt

5/11/20

Outline:

1. Recap
2. Controllability necessary and sufficient conditions
3. Observability definition
4. " nec. and suff. cond.
5. Reachability and pseudo-states.
6. i.d. zeros and reachability
7. Observability & o.d. zeros
8. Implication of coprimeness

$\Longrightarrow$ next topic realizations

$\boxed{1. \Longleftrightarrow 2.}$

$$\int_{\tau=0}^{t} e^{-A(t-\tau)} BB^T e^{-A^T(t-\tau)}\, d\tau$$

$\bar{\tau} = t-\tau$

$$\int_{\bar{\tau}=t}^{0} e^{-A\bar{\tau}} BB^T e^{A^T\bar{\tau}} (-d\bar{\tau})$$

$$= \int_0^t e^{\tau} BB^T e^{A^T\tau}\, d\tau$$

$\Longrightarrow \quad W_c \geq 0$

if is $W_c > 0 \Longleftrightarrow |W_c| \neq 0$

(W_c is nonsingular)

$|W_c| \neq 0 \Longrightarrow$ controllability

$$\boxed{x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau)\, d\tau}$$

We claim that the input $\quad x_1 = x(t_1)$

$$u(t) = -B^T e^{A(t_1-t)} W_c^{-1}\left[e^{At_1}x_0 - x_1\right]$$

achieves transfer from x_0 to x_1.

$t = t_1$

$$e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}(-BB^T)e^{A^T(t_1-\tau)}\, d\tau\; W_c^{-1}\left(e^{At_1}x_0 - x_1\right)$$

$$= e^{At_1}x_0 - W_c W_c^{-1}\left(e^{At_1}x_0 - x_1\right) = x_1$$

$$\boxed{\text{controllable} \Longrightarrow |W_c| \neq 0}$$

$$A \Longrightarrow B \quad \equiv \quad \overline{B} \Longrightarrow \overline{A}$$

Let us suppose that W_c is not positive definite for some time t_1.

There exists a $V \in \mathbb{R}^n$, such that

$$V^T W_c V = 0 = \int_0^{t_1} \underline{V^T e^{A\tau}} B B^T e^{A^T\tau} V \, d\tau$$

$$= \int_0^{t} \| B^T e^{A\tau} V \|^2 d\tau = 0$$

it implies $\quad B^T e^{A^T(t_1-\tau)} V \equiv 0$

or $\quad V^T e^{A(t_1-\tau)} B \equiv 0$

for all time $t \in [0; t_1]$

If it is controllable, there exists an input that transfers the state $X(0) = e^{-At_1} V$ to $X(t_1) = 0$

$$0 = V + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) \, d\tau$$

premultiplication with V^T gives

$$0 = v^T v + \int_0^{t_1} \underbrace{v^T e^{A(t_1-\tau)} B}_{=0} u(\tau)\, d\tau$$

$$0 = \|v\|^2$$

which contradicts $v \neq 0$

$\boxed{2. \Longleftrightarrow 3.}$

Because $W_c(t)$ nonsingular

$$\Longleftrightarrow$$

$$\not\exists\, v \quad \text{such that } v^T e^{At} B = 0$$

$$W_c(t) \text{ nonsingular} \underset{(2. \Rightarrow 3.)}{\Longrightarrow} \mathcal{C}(A,B) \text{ full rank}$$

$$A \Rightarrow B$$

$$\equiv$$

$$\overline{B} \Rightarrow \overline{A}$$

Let us suppose that $\mathcal{C}(A,B)$ is not full rank

$$\exists\, v \neq 0 \quad v^T \mathcal{C} = 0 \quad v^T A^k B = 0$$

$$k = 0, 1, 2, \ldots, n-1$$

$$\Rightarrow \quad v^T e^{At} B = 0$$

because at each t e^{At} can be expressed as a linear combination of $I, A, \dots, A^{n-1}$

$e^{At}B$ lin. comb. $[B \;\; AB \;\; \dots \;\; A^{n-1}B]$

$\Rightarrow$ W_c should be singular.

$\mathcal{C}(A,B) \Rightarrow W_c$ nonsingular

(3. $\Rightarrow$ 2.)

Suppose W_c singular, i.e. $\exists\, v \neq 0$

$$v^T e^{At} B = 0, \quad \forall t$$

Take $t=0$ $\Rightarrow$ $v^T B = 0$

differentiating $v^T e^{At} B = 0$ and setting $t=0$ $\quad v^T AB = 0$

differentiating again and setting $t=0$ $\quad v^T A^2 B = 0 \;\dots\; v^T A^{n-1} B = 0$

$$v^T\, \mathcal{C}(A,B) = 0.$$

$\boxed{3. \Leftrightarrow 4.}$ $\quad$ 3. $\Rightarrow$ 4.

$\mathcal{C}(A,B)$ full rank $\Rightarrow$ $[A - \lambda_i I \;\vdots\; B]$ full rank

λ_i eigenvalue of A

$A \Rightarrow B$

let us show $\overline{B} \Rightarrow \overline{A}$

Suppose $\left[A - \lambda_i I \;\vdots\; B \right]$ not full rank

i.e. $\exists q \neq 0 \quad q\,[A-\lambda_i I \;\; B] = [0 \;\; 0]$

which implies $\quad q A = \lambda_i q$ and $\underline{\underline{qB = 0}}$

$$q A^2 = (qA)A = \lambda_i^2 q$$

$$q\,[B \;\; AB \;\cdots\; A^{n-1}B] =$$

$$[\underbrace{qB}_{0} \;\; \lambda_i \underbrace{qB}_{0} \;\cdots\; \lambda_i^{n-1}\underbrace{qB}_{0}] = 0.$$

Since $q \neq 0$ $\quad \mathcal{C}(A,B)$ is not full rank.

$$\operatorname{rank}\left(\mathcal{C}(A,B)\right) < n \quad \Longrightarrow \operatorname{rank}\left([A-\lambda_i I \;\; B]\right) < n$$

<u>Lemma</u>

Let us suppose the system controllable

controllability is invariant by similarity equivalence

$$\overline{A} = P A P^{-1} \qquad \overline{B} = PB$$

$$\operatorname{rank} \mathcal{C}(A,B) = \operatorname{rank} \mathcal{C}(\overline{A},\overline{B}).$$

Theorem: $\quad \operatorname{rank}(\mathcal{C}(A,B)) = n_1 < n$

form the $n \times n$ matrix

$$P^{-1} = \begin{bmatrix} q_1 & \cdots & q_{n_1} & \cdots & q_n \end{bmatrix}$$

where the first n_1 columns are any n_1 linearly independent columns of $\mathcal{C}(A,B)$, and the remaining columns can be chosen as long as P is nonsingular.

$$\bar{x} = Px$$
$$x = P^{-1}\bar{x}$$

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

$$\bar{D} = D.$$

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \qquad \bar{B} = PB = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

$\bar{A}_{\bar{c}}$ is an $m \times m$ matrix

Let λ_1 be an eigenvalue of $\bar{A}_{\bar{c}}$

$$q_1 \bar{A}_{\bar{c}} = \lambda_1 q_1 \qquad q_1 \in \mathbb{R}^{1 \times m}$$

$$q_1 \left(\bar{A}_{\bar{c}} - \lambda_1 I \right) = 0$$

Construct the $1 \times n$

$$q \triangleq [\underbrace{0}_{n-m} \quad \underbrace{q_1}_{m}]$$

$$q \begin{bmatrix} \bar{A} - \lambda_1 I & \bar{B} \end{bmatrix} = \begin{bmatrix} 0 & q_1 \end{bmatrix} \begin{bmatrix} \bar{A}_c - \lambda_1 I & \bar{A}_{12} & \bar{B}_c \\ 0 & \bar{A}_{\bar{c}} - \lambda_1 I & 0 \end{bmatrix}$$

$$= 0$$

which implies $\text{rank}\left([\bar{A} - \lambda I \,\vdots\, \bar{B}] \right) < n$

and consequently $\text{rank}\left([A - \lambda I \,\vdots\, B] \right) < n$

for some eigenvalue λ of A.

$\boxed{2. \longleftrightarrow 5.}$

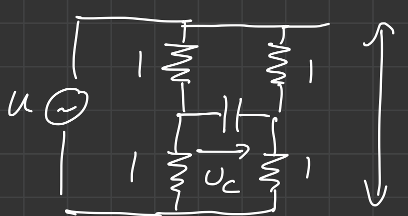
(We will come back to this during the stability and stabilization lecture)

it requires the solution to a Lyapunov equation.

Observability:

The system
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (*)$$

is said to be observable if for any unknown initial condition $x(0)$, there exists a finite $t_1 > 0$ such that knowledge of the input and output y over $[0; t_1]$, suffices to determine uniquely $x(0)$. Otherwise the system is said to be unobservable.



unobservable network

since if $u = 0$
it is impossible using y
to determine U_c

Response:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$Ce^{At}x_0 \triangleq \bar{y}(t)$$

$$\bar{y} = y - Du - C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

We must solve this equation for a fixed t.

in order to determine x_0 we need to wait, or use the time interval $[t_0; t_1]$.

<u>Theorem:</u> The system ⊛ is observable

$$\Longleftrightarrow$$

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.

<u>Proof:</u> $C e^{At} x(0) = \bar{y}(t)$ $\quad t_0 = 0.$

$W_c(t_1)^{-1}$ exists $\Longrightarrow$ observable

$$\int_0^{t_1} \left(e^{A^T\tau} C^T C e^{A\tau} \right) x(0) =$$

$$\int_0^{t_1} e^{A^T\tau} C^T \bar{y}(\tau) d\tau$$

if $W_o(t_1)$ is nonsingular then

$$x_0 = W_o(t_1)^{-1} \int_0^{t_1} e^{A^T\tau} C^T \bar{y}(\tau) d\tau$$

This yields a unique x_0, i.e. observability is verified.

observable $\Longrightarrow$ $W_o(t_1)^{-1}$ exists

Suppose that $W_o(t_1)$ is singular, i.e. only positive semi-definite, $\exists\, v$

$$v^T W_o(t_1) v = 0 = \int_0^{t_1} \underbrace{v^T e^{A^T\tau} C^T}\, \underbrace{C e^{A\tau} v}\, d\tau$$

$$\int_0^{t_1} \| C e^{A\tau} v \|^2 \, d\tau = 0$$

which implies $\quad C e^{A\tau} v \equiv 0$

for all τ in $[0, t_1]$

then $\quad x_1(0) = v \neq 0$

$x_2(0) = 0$

(two different initial conditions $x(0) = x_1(0)$ and $x(0) = x_2(0)$)

both yield

$$y(t) = C e^{At} x_i(0) = 0$$

$(u = 0)$

the system is unobservable.

Duality Theorem

(A, B) controllable

$$\Longleftrightarrow \quad (A^T \; B^T) \text{ observable}$$

$$\boxed{\dot{x} = Ax + Bu}$$

$y = \ldots$

controllable

$$\boxed{\begin{aligned} \dot{x} &= A^T x \\ y &= B^T x \end{aligned}} + \ldots u$$

observable

1. $W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau$

   is nonsingular for any $t > 0$.

2. $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ C A^{n-1} \end{bmatrix}$

   $\text{rank}(\mathcal{O}) = n$

   $\text{rank } \mathcal{O} = \text{rank}\left(\left[C^T \; A^T C^T \; \cdots \; (A^T)^{n-1} C^T\right]\right)$

3. $\begin{bmatrix} A - \lambda_i I \\ \hline C \end{bmatrix}$ has full column rank at every eigenvalue λ_i of A.

4. If in addition all eigenvalues of A have negative real parts, then the unique solution to $A^T W_o + W_o A = -C^T C$ is positive definite

$$W_o = \int_0^\infty e^{A^T \tau} C C^T e^{A\tau} \, d\tau$$

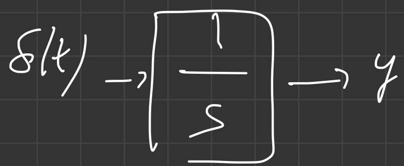
Either one of 1. to 4. are necessary and sufficient conditions for observability.

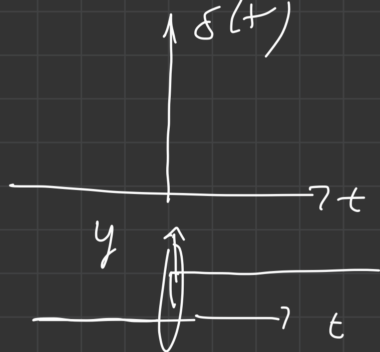
<u>Reachability</u> pseudo-state trajectory $\xi(t)$

$$\begin{cases} T\left(\frac{d}{dt}\right)\xi + U\left(\frac{d}{dt}\right)u = 0 \\ -V\left(\frac{d}{dt}\right)\xi + W\left(\frac{d}{dt}\right)u = y \end{cases}$$

$$\begin{cases} T(s)\xi + U(s)u = 0 \\ -V(s)\xi + W(s)u = y. \end{cases}$$

<u>Definition</u>: A zero input pseudo-state trajectory $\{\xi(t)\}$ is called <u>reachable</u> if and only if an input of the form $\sum_{\alpha=0}^{m} u_\alpha \{\delta^{(\alpha)}(t)\}$





where $\{\delta(t)\}$ is the Dirac distribution which produces a pseudo state trajectory $\{\xi_1(t)\}$ such

$t > 0$ $$\xi_1(t) = \xi(t)$$

u kicks the system in a state such that the pseudo-state follows the desired one.

Thm: Every zero-input pseudo-state $\{\xi(t)\}$ is reachable

$$\Longleftrightarrow$$

(T, U) is left coprime

Corollary: Every zero-input pseudo-state ξ solution to $T\left(\frac{d}{dt}\right)\xi = 0$ is reachable $\Longleftrightarrow$ $P(s)$ has no input decoupling zeros.

Definition: A zero-input pseudo-state trajectory of $T\left(\frac{d}{dt}\right)\xi$ is said to be unobservable if and only if the corresponding zero-input response substitute

$$y(t) = -V\left(\frac{d}{dt}\right)\xi(t) = 0$$

Thm $\{\xi(t)\}$ is unobservable

$$\Longleftrightarrow$$

$$R\left(\frac{d}{dt}\right)\xi = 0$$

where R is a g.c.r.d of

$$\begin{pmatrix} -U & W \end{pmatrix}$$

Thm: $\xi(t)$ observable

$$\Longleftrightarrow$$

$P(s)$ has no output decoupling zeros.

$\Rightarrow$ next week implications of coprimeness from the classical state-space perspective

$\Rightarrow$ realizations. balancing between controllability and obs.